## Grupa A, Pismeni ispit iz Matematike II, 27.01.2014.
## ispit pisati isključivo hemijskom olovkom

**1.** Isključivo primjenom dvostrukog integrala izračunati površinu figure koju prave linije $y^2 = x$, $y^2 = 4x$ i $x = 4$.

**2.** Izračunati površinu dijela konusa $y^2 = 2xz$ koji se nalazi u prvom oktantu između površina $x = 1$ i $z = 3$.

**3.** Metodom diferenciranja po parametru izračunati integral $\displaystyle\int\limits_0^\infty \frac{\operatorname{arc\,tg} \alpha x}{x(1+x^2)} \mathrm{d}x \quad (\alpha > 0)$.

**4.** Izračunati cirkulaciju vektorskog polja $\vec{v} = (x-2)\vec{i} + (x+y)\vec{j} - 2z\vec{k}$ duž stranica trougla $\triangle ABC$ čiji su vrhovi $A(1;0;0)$, $B(0;1;0)$ i $C(0;0;1)$.

## Grupa B, Pismeni ispit iz Matematike II, 27.01.2014.
## ispit pisati isključivo hemijskom olovkom

**1.** Isključivo primjenom dvostrukog integrala izračunati površinu figure koju prave linije $y^2 = -x$, $y^2 = -9x$ i $x = -9$.

**2.** Izračunati površinu dijela konusa $z^2 = 2xy$ koji se nalazi u prvom oktantu između površina $x = 2$ i $y = 4$.

**3.** Metodom diferenciranja po parametru izračunati integral $\displaystyle\int\limits_0^\infty \frac{1 - e^{-\alpha x^2}}{x e^{x^2}} \mathrm{d}x \quad (\alpha > -1)$.

**4.** Izračunati cirkulaciju vektorskog polja $\vec{v} = (xz, -yz^2, xy)$ duž zatvorene linije $L : \begin{cases} z = x^2 - y^2 + 2a^2 \\ x^2 + y^2 = a^2 \end{cases}$.


## Grupa A, Pismeni ispit iz Matematike II, 27.01.2014.
## ispit pisati isključivo hemijskom olovkom

**1.** Isključivo primjenom dvostrukog integrala izračunati površinu figure koju prave linije $y^2 = x$, $y^2 = 4x$ i $x = 4$.

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**4.** Izračunati cirkulaciju vektorskog polja $\vec{v} = (x-2)\vec{i} + (x+y)\vec{j} - 2z\vec{k}$ duž stranica trougla $\triangle ABC$ čiji su vrhovi $A(1;0;0)$, $B(0;1;0)$ i $C(0;0;1)$.

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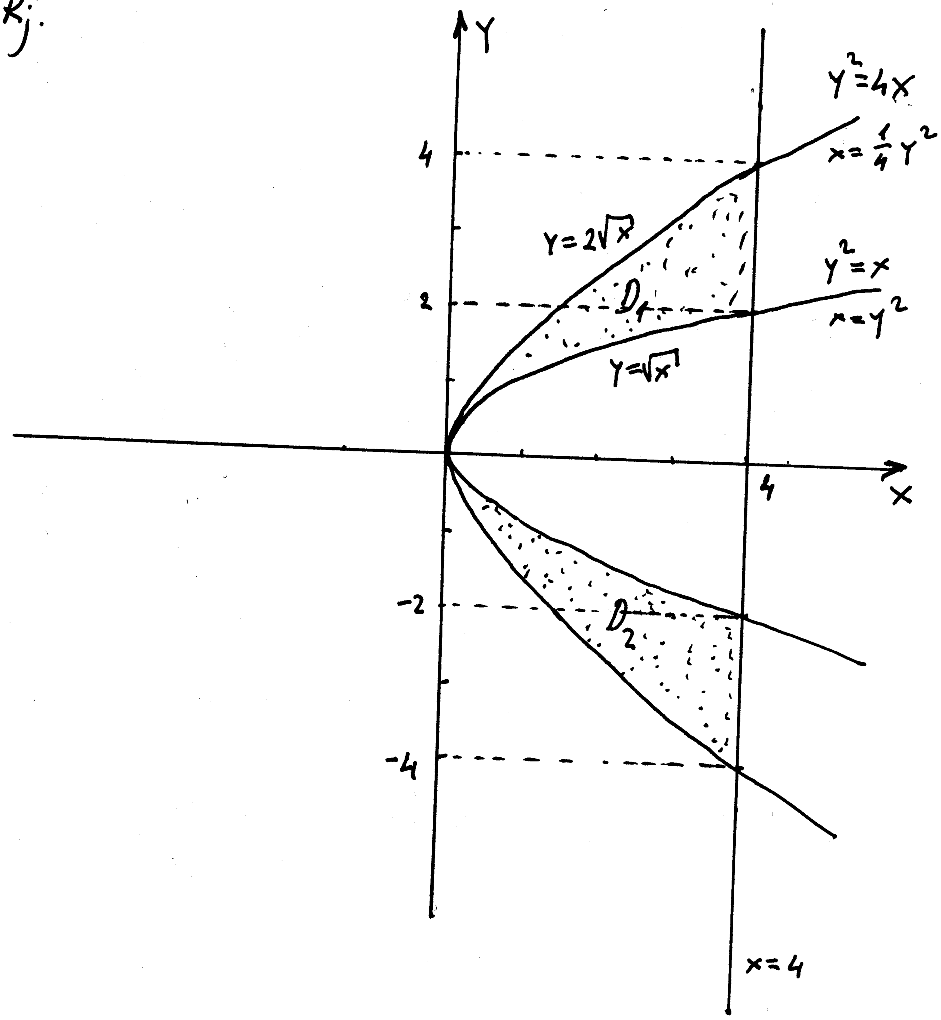
Za uočene greške pisati na *infoarrt@gmail.com*

(#) Isključivo primjenom dvostrukog integrala izračunati površinu figure koju prave linije $y^2=x$, $y^2=4x$ i $x=4$.

Rj.




$$P=\iint_{D_1} dx\,dy+\iint_{D_2} dx\,dy$$

$$=2\iint_{D_1} dx\,dy$$

$$\iint_{D_1} dx\,dy=\int_0^4 dx\int_{\sqrt{x}}^{2\sqrt{x}} dy=$$

$$=\int_0^4 \sqrt{x}\,dx=\frac{16}{3}$$

Prema tome $P=\frac{32}{3}$

II način – prvo po y-nu

$$\int_0^2 dy\int_{\frac{1}{4}y^2}^{y^2} dx+\int_2^4 dy\int_{\frac{1}{4}y^2}^{4} dx=\int_0^2 \frac{3}{4}y^2\,dy+\int_2^4\left(4-\frac{1}{4}y^2\right)dy=2+\frac{10}{3}=\frac{16}{3},$$

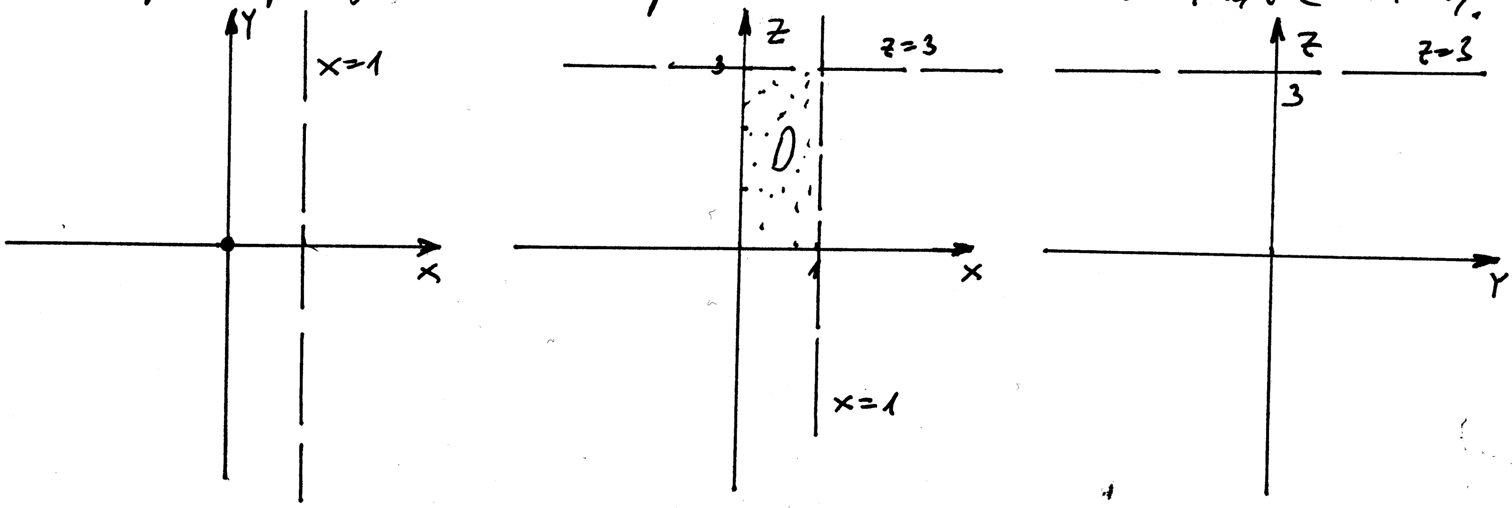
$$P=\frac{32}{3}$$

# Izračunati površinu dijela konusa $y^2=2xz$ koji se nalazi u prvom oktantu između površina $x=1$ i $z=3$.

Rj.

Površinu oblasti $S$ računamo po formuli $P=\iint\limits_S dS$

Skicirajmo presjeke datih površina sa tri koordinatne ravni.




U prvom oktantu tražimo površinu dijela figure $y=\sqrt{2}\sqrt{xz}$.

$$y=\sqrt{2}\sqrt{x}\sqrt{z},\quad y'_x=\sqrt{2}\,\frac{\sqrt{z}}{2\sqrt{x}},\quad y'_z=\sqrt{2}\,\frac{\sqrt{x}}{2\sqrt{z}}$$

$$\sqrt{1+z_x'^2+z_y'^2}=\sqrt{1+\frac{z}{2x}+\frac{x}{2z}}=\frac{1}{\sqrt{2}}\sqrt{2+\frac{z}{x}+\frac{x}{z}}=\frac{1}{\sqrt{2}}\sqrt{\frac{2xz+z^2+x^2}{xz}}=$$

$$=\frac{1}{\sqrt{2}}\sqrt{\frac{(x+z)^2}{xz}}=\frac{1}{\sqrt{2}}\cdot\frac{|x+z|}{\sqrt{xz}}\ \underset{|x+z|=x+z}{\overset{\text{I oktant}}{=\!=}}\ \frac{1}{\sqrt{2}}\left(\frac{x}{\sqrt{xz}}+\frac{z}{\sqrt{xz}}\right)=\frac{1}{\sqrt{2}}\left(\frac{\sqrt{x}}{\sqrt{z}}+\frac{\sqrt{z}}{\sqrt{x}}\right)$$

$$P=\iint\limits_S dS=\iint\limits_D\frac{1}{\sqrt{2}}\left(\frac{\sqrt{x}}{\sqrt{z}}+\sqrt{\frac{z}{x}}\right)dx\,dz=\frac{1}{\sqrt{2}}\int_0^1 dx\int_0^3\left(\sqrt{x}\,z^{-\frac{1}{2}}+\frac{1}{\sqrt{x}}\,z^{\frac{1}{2}}\right)dz$$

$$=\ldots=\frac{1}{\sqrt{2}}\int_0^1 2\sqrt{3}\left(\sqrt{\frac{1}{x}}+\sqrt{x}\right)dx=\ldots=\frac{1}{\sqrt{2}}\cdot 16\,\frac{\sqrt{3}}{3}=\frac{16\sqrt{3}}{3\sqrt{2}}=\frac{8\sqrt{6}}{3}$$

(#) Metodom diferenciranja po parametru izračunati integral

$$\int_0^\infty \frac{\operatorname{arc\,tg} \alpha x}{x(1+x^2)}\,dx \quad (\alpha > 0).$$

Rj.-upute:

Ako je $I(\alpha)=\int_a^b f(\alpha,x)\,dx$ tada $I'(\alpha)=\int_a^b f'_\alpha\,dx$

$$f(\alpha,x)=\frac{\operatorname{arc\,tg}\alpha x}{x(1+x^2)} \Rightarrow f'_\alpha=\frac{1}{x(1+x^2)}\cdot\frac{x}{1+\alpha^2x^2}$$

$$I(\alpha)=\int_0^\infty \frac{\operatorname{arc\,tg}\alpha x}{x(1+x^2)}\,dx \Rightarrow I'(\alpha)=\int_0^\infty\frac{dx}{(1+x^2)(1+\alpha^2x^2)}.$$

$$\frac{1}{(1+x^2)(1+\alpha^2x^2)}=\frac{Ax+B}{1+x^2}+\frac{Cx+D}{1+\alpha^2x^2} \Rightarrow \dots \Rightarrow \begin{array}{l} A=0 \\ B=\dfrac{-1}{\alpha^2-1} \\ C=0,\ D=\dfrac{\alpha^2}{\alpha^2-1}\end{array}$$

$$I'(\alpha)=\int_0^\infty\frac{\frac{-1}{\alpha^2-1}}{1+x^2}\,dx+\int_0^\infty\frac{\frac{\alpha^2}{\alpha^2-1}}{\alpha^2\left(\frac{1}{\alpha^2}+x^2\right)}\,dx=-\frac{1}{\alpha^2-1}\operatorname{arc\,tg}x\Big|_0^\infty+\frac{\alpha^2}{\alpha^2-1}\frac{\alpha}{\alpha^2}\operatorname{arc\,tg}\alpha x\Big|_0^\infty$$

$$=\frac{-1}{\alpha^2-1}\cdot\frac{\pi}{2}+\frac{\alpha}{\alpha^2-1}\cdot\frac{\pi}{2}=\frac{\pi}{2}\frac{\alpha-1}{(\alpha-1)(\alpha+1)}=\frac{\pi}{2}\cdot\frac{1}{\alpha+1} \Rightarrow I(\alpha)=\frac{\pi}{2}\ln|\alpha+1|+C$$

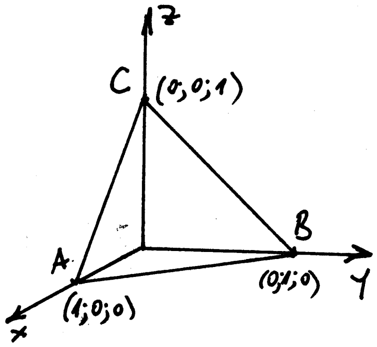
Kako je $I(0)=0=\frac{\pi}{2}\ln 1+C=C \Rightarrow C=0$ pa je za $\alpha>0$

$$\int_0^\infty\frac{\operatorname{arc\,tg}\alpha x}{x(1+x^2)}\,dx=\frac{\pi}{2}\ln|\alpha+1|.$$

\# Izračunati cirkulaciju vektorskog polja $\vec{v} = (x-2)\vec{i} + (x+y)\vec{j} - 2z\vec{k}$ duž stranica trougla $\Delta ABC$ čiji su vrhovi $A(1;0;0)$, $B(0;1;0)$ i $C(0;0;1)$.

Rj. Cirkulaciju vektorskog polja računamo po formuli

$$C = \int_C \vec{v}\, d\vec{r} = \int_C v_x\,dx + v_y\,dy + v_z\,dz = \int_{ABCA} (x-2)\,dx + (x+y)\,dy + (-2z)\,dz$$




$\overset{x_1\,y_1\,z_1}{A(1;0;0)}$
$B(0;1;0)$ $\quad p(A,B):\ \frac{x-1}{0-1} = \frac{y-0}{1-0} = \frac{z-0}{0}$

$-x+1 = y, \quad z=0$

$y = -x+1 \qquad dy = -dx$

$y$ uzima vrijednosti od 1 do 0

$$\int_{AB} (x-2)\,dx + (x+y)\,dy - 2z\,dz = \int_1^0 \underbrace{\left[(x-2) + (x-x+1)(-1)\right]}_{x-3} dx = \int_1^0 (x-3)\,d(x-3) = \frac{1}{2}(x-3)^2\Big|_1^0 \underset{9-4}{} = \frac{5}{2}$$

$\overset{x_1\,y_1\,z_1}{B(0;1;0)}$
$C(0;0;1)$ $\quad p(B,C):\ \frac{x-0}{0} = \frac{y-1}{-1} = \frac{z-0}{1}$

$x=0, \quad -y+1 = z$ $\qquad$ z uzima vrijednosti od 1 do 0

$z = -y+1$ $\qquad dx=0,\ dz = -dy$

z uzima vrijednosti od

$$\int_{BC} = \int_1^0 \underbrace{(y - 2(-y+1)(-1))}_{y-2y+2}\,dz = \int_1^0 (-y+2)(-1)\,d(-y+2) = \frac{1}{2}(-y+2)^2\Big|_0^1 \underset{1-4}{} = -\frac{3}{2}$$

$\overset{x_1\,y_1\,z_1}{C(0;0;1)}$
$A(1;0;0)$ $\Rightarrow p(C,A):\ \frac{x-0}{1} = \frac{y-0}{0} = \frac{z-1}{-1} \Rightarrow x = -z+1$ gdje $z\Big|_1^0$

$dx = -dz$

$$\int_{CA} = \int_1^0 \left[\underbrace{(-z+1-2)(-1)}_{z+1} - 2z\right] dz = \int_1^0 (-z+1)(-1)\,d(-z+1) = \frac{1}{2}(-z+1)^2\Big|_0^1 \underset{(0-1)}{} = -\frac{1}{2}$$

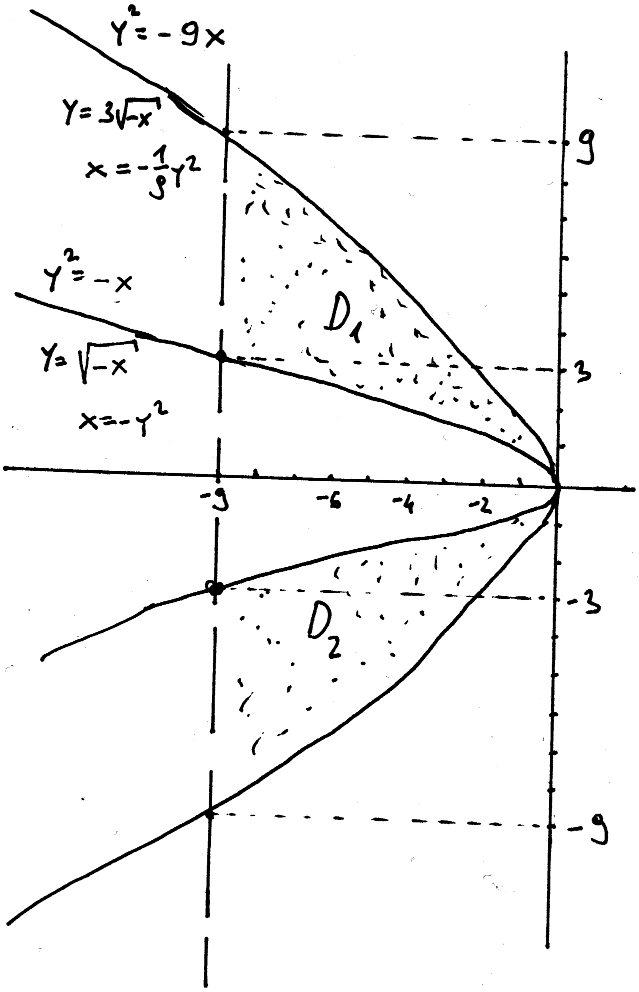
Prema tome

KONAČNO RJEŠENJE JE:

$$C = \oint_{ABCA} = \int_{AB} + \int_{BC} + \int_{CA} = \boxed{\frac{1}{2}}$$

(#) Isključivo primjenom dvostrukog integrala izračunati površinu figure koju prave linije $y^2=-x$, $y^2=-9x$ i $x=-9$.

Rj.




$$P=\iint_{D_1}dx\,dy+\iint_{D_2}dx\,dy=2\iint_{D_1}dx\,dy$$

$$\iint_{D_1}dx\,dy=\int_{-9}^{0}dx\int_{\sqrt{-x}}^{3\sqrt{-x}}dy=\int_{-9}^{0}2\sqrt{-x}\,dx$$

$$=36$$

$$P=2\cdot 36=72$$

II način – prvo po y-nu

$$\int_0^3 dy\int_{-y^2}^{-\frac{1}{9}y^2}dx+\int_3^9 dy\int_{-9}^{-\frac{1}{9}y^2}dx=\int_0^3\frac{8}{9}y^2\,dy+\int_3^9\left(9-\frac{1}{9}y^2\right)dy=8+28=36$$

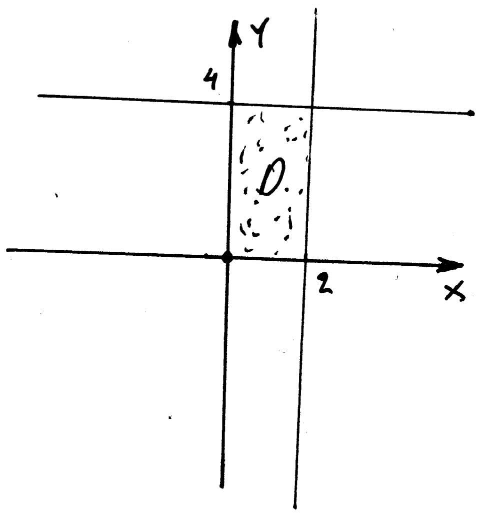
$$P=2\cdot 36=72$$

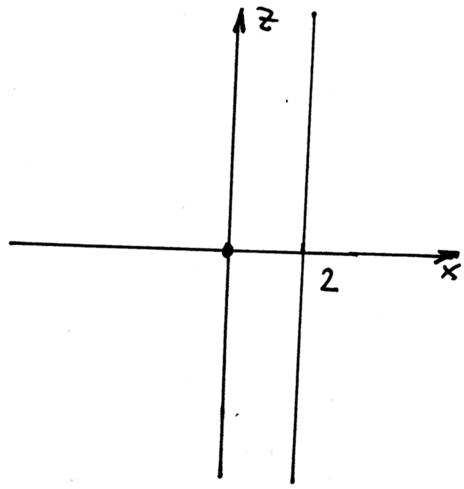
⊕ Izračunati površinu dijela konusa $z^2 = 2xy$ koji se nalazi u prvom oktantu između površina $x=2$ i $y=4$.

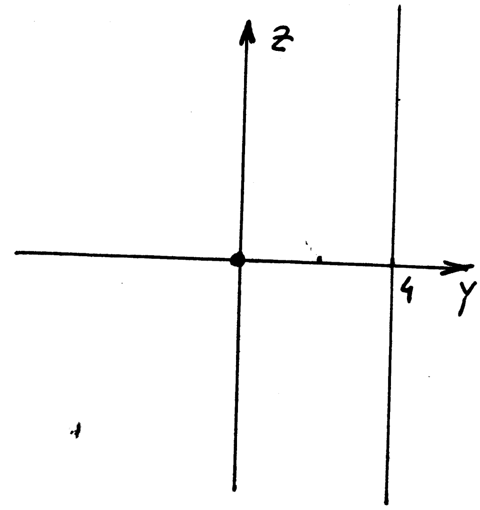
Rj.

Površina oblasti $S$ se računa po formuli $P = \iint\limits_S dS$.

Skicirajmo presjeke datih površina sa tri koordinatne ravni.




Kako je u pitanju prvi oktant to je $z \geq 0$, a tražimo površinu figure $z = \sqrt{2}\sqrt{xy}$ $\left( \Rightarrow z'_x = \sqrt{2}\frac{\sqrt{y}}{2\sqrt{x}}, \; z'_y = \sqrt{2}\frac{\sqrt{x}}{2\sqrt{y}} \right)$

$$P = \iint\limits_S dS = \iint\limits_D \sqrt{1 + {z'_x}^2 + {z'_y}^2}\, dx dy = \iint\limits_D \sqrt{1 + \frac{y}{2x} + \frac{x}{2y}}\, dx dy =$$

$$= \frac{1}{\sqrt{2}} \iint\limits_D \sqrt{\frac{2xy + x^2 + y^2}{xy}}\, dx dy = \frac{1}{\sqrt{2}} \int_0^2 dx \int_0^4 \left( \sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \right) dy =$$

$\boxed{x^2 + 2xy + y^2 = (x+y)^2}$

$$= \frac{1}{\sqrt{2}} \int_0^2 \left( 2\sqrt{xy} + \frac{2}{3}\sqrt{\frac{y^3}{x}} \right) \Bigg|_{y=0}^{y=4} dx =$$

$$= 2\sqrt{2} \int_0^2 \left( \sqrt{x} + \frac{4}{3\sqrt{x}} \right) dx = 2\sqrt{2} \left( \frac{2}{3}\sqrt{x^3} + \frac{8}{3}\sqrt{x} \right) \Bigg|_0^2 = 16$$

# Metodom diferenciranja po parametru izračunati integral

$$\int_0^\infty \frac{1-e^{-\alpha x^2}}{xe^{x^2}}\,dx \quad (\alpha > -1).$$

Rj.

Ako je $I(\alpha) = \int_a^b f(\alpha, x)\,dx$ tada $I'(\alpha) = \int_a^b f'_\alpha \,dx$.

$$f(\alpha, x) = \frac{1-e^{-\alpha x^2}}{xe^{x^2}} \Rightarrow f'_\alpha = \frac{1}{xe^{x^2}}(-1)e^{-\alpha x^2}\cdot(-x^2) = \frac{xe^{-\alpha x^2}}{e^{x^2}}$$

$$I(\alpha) = \int_0^\infty \frac{1-e^{-\alpha x^2}}{xe^{x^2}}\,dx \Rightarrow I'(\alpha) = \int_0^\infty \frac{xe^{-\alpha x^2}}{e^{x^2}}\,dx =$$

$$= \int_0^\infty x e^{-\alpha x^2 - x^2}\,dx = \int_0^\infty x e^{-(\alpha+1)x^2}\,dx = \left| \begin{array}{ll} -(\alpha+1)x^2 = u & \\ -2(\alpha+1)x\,dx = du & x\Big|_0^\infty \Rightarrow u\Big|_0^{-\infty} \\ x\,dx = \frac{-1}{2(\alpha+1)}\,du & \end{array} \right|$$

$$= \frac{1}{2(\alpha+1)}\int_{-\infty}^0 e^u\,du = \frac{1}{2(\alpha+1)} \overbrace{e^u\Big|_{-\infty}^0}^{1-e^{-\infty}} = \frac{1}{2(\alpha+1)} \Rightarrow I'(\alpha) = \frac{1}{2}\cdot\frac{1}{\alpha+1}$$

$$\Rightarrow I(\alpha) = \frac{1}{2}\ln|\alpha+1| + C \text{ ; A kako je } I(0) = \int_0^\infty \frac{\overset{1-e^0=1-1}{0}\,dx}{xe^{x^2}} = 0 = \frac{1}{2}\ln 1 + C$$

$$\Downarrow$$

$$C = 0$$

Prema tome
$$\int_0^\infty \frac{1-e^{-\alpha x^2}}{xe^{x^2}}\,dx = \frac{1}{2}\ln|\alpha+1|$$

(#) Izračunati cirkulaciju vektorskog polja $\vec{v} = (xz, -yz^2, xy)$ duž zatvorene linije $L$

$$L: \begin{cases} z = x^2 - y^2 + 2a^2 \\ x^2 + y^2 = a^2 \end{cases}$$

Rj:

Cirkulaciju vektorskog polja $\vec{v}$ računamo po formuli

$$C = \int_C \vec{v}\, d\vec{r} = \int_C v_x dx + v_y dy + v_z dz$$

a u našem slučaju $\quad C = \oint_C xz\, dx + (-yz^2)\, dy + xy\, dz$

Jedan od pristupa za rješavanje ovog zadatka je da parametrizujemo krivu $L$. Kako je u krivoj data kružnica $x^2+y^2=a^2$ prvo ćemo parametrizirati nju pa dvije dobijene vrijednosti uvrstiti u $z$.

$$\begin{aligned} x &= a\cos\varphi \\ y &= a\sin\varphi \\ 0 &\le \varphi \le 2\pi \end{aligned} \quad \Rightarrow \quad z = a^2\cos^2\varphi - a^2\sin^2\varphi + 2a = a^2(2+\cos 2\varphi)$$

$$dx = -a\sin\varphi, \quad dy = a\cos\varphi, \quad dz = -2a^2\sin 2\varphi$$

$$C = -\frac{a^4}{2}\int_0^{2\pi}(2+\cos 2\varphi)\sin 2\varphi\, d\varphi - \frac{a^6}{2}\int_0^{2\pi}(2+\cos 2\varphi)^2\sin 2\varphi\, d\varphi -$$

$$- a^4\int_0^{2\pi}\sin^2 2\varphi\, d\varphi = \frac{a^4}{8}(2+\cos 2\varphi)^2\Big|_0^{2\pi} + \frac{a^6}{12}(2+\cos 2\varphi)^3\Big|_0^{2\pi} -$$

$$-\frac{a^4}{2}\left(\varphi - \frac{1}{4}\sin 4\varphi\right)\Big|_0^{2\pi} = -\pi a^4$$